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EXPLICIT CRITERIA FOR SEVERAL TYPES OF ERGODICITY

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Abstract. The explicit criteria, collected in Tables 5.1 and 5.2, for several types of ergodicity of one-dimensional diffusions or birth-death processes have been found out recently in a surprisingly short period. One of the criteria is for exponential ergodicity of birth-death processes. This problem has been opened for a long time in the study of Markov chains. The survey article explains in details the idea which leads to solve the problem just mentioned. It is interesting that the problem is connected with several branches of mathematics. Some open problems for the further study are also proposed.

Let us begin with the paper by recalling the three traditional types of ergodicity.

1. Three traditional types of ergodicity. Let $Q = (q_{ij})$ be a regular Q-matrix on a countable set $E = \{i, j, k, \cdots\}$. That is, $q_{ij} \geq 0$ for all $i \neq j$, $q_i := -q_{ii} = \sum_{j\neq i} q_{ij} < \infty$ for all $i \in E$ and Q determines uniquely a transition probability matrix $P(t) = (p_{ij}(t))$ (which is also called a Q-process or a Markov chain). Denote by $\pi = (\pi_i)$ a stationary distribution of $P(t)$: $\pi P(t) = \pi$ for all $t \geq 0$. From now on, assume that the Q-matrix is irreducible and hence the stationary distribution π is unique. Then, the three types of ergodicity are defined respectively as follows.

$$\textit{Ordinary ergodicity}: \quad \lim_{t\to\infty} |p_{ij}(t) - \pi_j| = 0 \tag{1.1}$$

$$\textit{Exponential ergodicity}: \quad \lim_{t\to\infty} e^{\hat{\alpha} t}|p_{ij}(t) - \pi_j| = 0 \tag{1.2}$$

$$\begin{aligned} \textit{Strong ergodicity}: \quad &\lim_{t\to\infty} \sup_i |p_{ij}(t) - \pi_j| = 0 \\ &\Longleftrightarrow \lim_{t\to\infty} e^{\hat{\beta} t} \sup_i |p_{ij}(t) - \pi_j| = 0, \end{aligned} \tag{1.3}$$

where $\hat{\alpha}$ and $\hat{\beta}$ are (the largest) positive constants and i, j varies over whole E. The definitions are meaningful for general Markov processes once the pointwise convergence is replaced by the convergence in total variation norm. The three types of ergodicity were studied in a great deal during 1953–1981. Especially, it was proved that strong ergodicity $\Longrightarrow$ exponential ergodicity $\Longrightarrow$ ordinary ergodicity. Refer to Anderson (1991), Chen (1992, Chapter 4) and Meyn and Tweedie (1993) for details and related references. The study is quite complete in the sense that we have the following criteria which are described by the Q-matrix plus a test sequence (y_i) only.

Theorem 1.1 (Criteria). *Let $H \neq \emptyset$ be an arbitrary but fixed finite subset of E. Then the following conclusions hold.*

(1) *The process $P(t)$ is ergodic iff the system of inequalities*

$$\begin{cases} \sum_j q_{ij} y_j \leq -1, & i \notin H \\ \sum_{i\in H} \sum_{j\neq i} q_{ij} y_j < \infty \end{cases} \tag{1.4}$$

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has a nonnegative finite solution (y_i).

(2) *The process* $P(t)$ *is exponentially ergodic iff for some* $\lambda > 0$ *with* $\lambda < q_i$ *for all* i, *the system of inequalities*
$$\begin{cases} \sum_j q_{ij} y_j \le -\lambda y_i - 1, \quad i \notin H \\ \sum_{i\in H}\sum_{j\ne i} q_{ij} y_j < \infty \end{cases} \tag{1.5}$$
has a nonnegative finite solution (y_i).

(3) *The process* $P(t)$ *is strongly ergodic iff the system (1.4) of inequalities has a bounded nonnegative solution* (y_i).

The probabilistic meaning of the criteria reads respectively as follows:
$$\max_{i\in H} \mathbb{E}_i \sigma_H < \infty, \quad \max_{i\in H} \mathbb{E}_i e^{\lambda\sigma_H} < \infty \quad \text{and} \quad \sup_{i\in E} \mathbb{E}_i \sigma_H < \infty,$$
where $\sigma_H = \inf\{t \ge \text{the first jumping time} : X_t \in H\}$ and λ is the same as in (1.5). The criteria are not completely explicit since they depend on the test sequences (y_i) and in general it is often non-trivial to solve a system of infinite inequalities. Hence, one expects to find out some explicit criteria for some specific processes. Clearly, for this, the first choice should be the birth-death processes. Recall that for a birth-death process with state space $E = \mathbb{Z}_+ = \{0, 1, 2, \cdots\}$, its Q-matrix has the form: $q_{i,i+1} = b_i > 0$ for all $i \ge 0$, $q_{i,i-1} = a_i > 0$ for all $i \ge 1$ and $q_{ij} = 0$ for all other $i \ne j$. Along this line, it was proved by Tweedie (1981)(see also Anderson (1991) or Chen (1992)) that
$$S := \sum_{n\ge 1} \mu_n \sum_{j\le n-1} \frac{1}{\mu_j b_j} < \infty \Longrightarrow \text{Exponential ergodicity},$$
where $\mu_0 = 1$ and $\mu_n = b_0 \cdots b_{n-1}/a_1 \cdots a_n$ for all $n \ge 1$. Refer to Wang (1980), Yang (1986) or Hou et al (2000) for the probabilistic meaning of S. However, the condition is not necessary. A simple example is as follows. Let $a_i = b_i = i^\gamma$ $(i \ge 1)$ and $b_0 = 1$. Then the process is exponential ergodic iff $\gamma \ge 2$ (see Chen (1996)) but $S < \infty$ iff $\gamma > 2$. Surprisingly, the condition is correct for strong ergodicity.

Theorem 1.2 (H. J. Zhang, Lin and Hou (1998)). $S < \infty \Longleftrightarrow$ *Strong ergodicity.*

With a different proof, the result is extended by Y. H. Zhang (2001) to the single-birth processes with state space $\mathbb{Z}_+$. Here, the term "single birth" means that $q_{i,i+1} > 0$ for all $i \ge 0$ but $q_{ij} \ge 0$ can be arbitrary for $j < i$. Introducing this class of Q-processes is due to the following observation: If the first inequality in (1.4) is replaced by equality, then we get a recursion formula for (y_i) with one parameter only. Hence, there should exist an explicit criterion for the ergodicity (resp. uniqueness, recurrence and strong ergodicity). For (1.5), there is also a recursion formula but now two parameters are involved and so it is unclear whether there exists an explicit criterion or not for the exponential ergodicity.

Note that the criteria are not enough to estimate the convergence rate $\hat\alpha$ or $\hat\beta$ (cf. Chen (2000a)). It is the main reason why we have to come back to study the well-developed theory of Markov chains. For birth-death processes, the estimation of $\hat\alpha$ was studied by Doorn in a book (1981) and in a series of papers (1985, 1987, 1991). Especially, the precise $\hat\alpha$ was computed out for four particular models. This work has been very helpful in the later study, as will be discussed in the next section. However, there is no explicit criterion for $\hat\alpha > 0$ ever appeared so far. The difficulty mainly comes from the complex of two parameters and as we will see in the next section, there is indeed an intrinsic reason for it. By the way, we point out that there is nearly no result about the estimation of $\hat\beta$.

Open problem 1. *How to estimate* $\hat\beta$*?*

2. The first (non-trivial) eigenvalue (spectral gap). The birth-death processes have a nice property—symmetrizability: $\mu_i p_{ij}(t) = \mu_j p_{ji}(t)$ for all i, j and $t \ge 0$. Then, the matrix Q can be regarded as a self-adjoint operator on the real L^2-space $L^2(\mu)$ with norm $\|\cdot\|$. In other words, one can use the L^2-theory, which was a starting point of Doorn (1981). Furthermore, one can study the

L^2-exponential convergence given below. Assuming that $\mu := \sum_i \mu_i < \infty$ and then setting $\pi_i = \mu_i/\mu$, we have L^2-space $L^2(\pi)$ with norm $\|\cdot\|$. Then, the convergence means that

$$\|P(t)f - \pi(f)\| \leq \|f - \pi(f)\| \leq e^{-\lambda_1 t} \tag{1.6}$$

for all $t \geq 0$, where $\pi(f) = \int f \mathrm{d}\pi$ and λ_1 is the first non-trivial eigenvalue of $(-Q)$ (cf. Chen (1992, Chapter 9)).

The estimation of λ_1 for birth-death processes was studied by Sullivan (1984), Liggett (1989) and Landim, Sethuraman and Varadhan (1996) (see also Kipnis & Lamdin (1999)). It was used as a comparison tool to handle the convergence rate for some interacting particle systems, which are infinite-dimensional Markov processes.

The present author came to this topic by comparing λ_1 with $\hat{\alpha}$, which is the first result in (2.1). This transfers all known results about $\hat{\alpha}$ to λ_1. Then, by using the coupling methods, the author obtained a variational formula given in the second line of (2.1).

Theorem 2.1.

$$\begin{aligned}\hat{\alpha} &= \lambda_1 \qquad [\mathit{Chen}(1991)]\\ &= \sup_{w\in\mathcal{W}} \inf_{i\geq 0} I_i(w)^{-1} \quad [\mathit{Chen}(1996)]\end{aligned} \tag{2.1}$$

where $\mathcal{W} = \{w : w_i \uparrow\uparrow, \pi(w) \geq 0\}$ *and* $I_i(w) = \dfrac{1}{\mu_i b_i (w_{i+1} - w_i)} \displaystyle\sum_{j\geq i+1} \mu_j w_j$

In view of Theorem 2.1, in order to obtain a criterion for the exponential ergodicity, one needs only to have a representative sequence (w_i). Note that the test sequence (w_i) used in the formula is indeed a mimic of the eigenvector of λ_1, and in general, the eigenvector can be very sensitive. Thus, it becomes a question from the above formula about the existence of a representative sequence (w_i) for justifying the positiveness of $\hat{\alpha}$. Let us now leave Markov chains for a while and turn to diffusions.

3. One-dimensional diffusions. As a parallel of birth-death process, we now consider the elliptic operator $L = a(x)\mathrm{d}^2/\mathrm{d}x^2 + b(x)\mathrm{d}/\mathrm{d}x$ on the half line $[0, \infty)$ with $a(x) > 0$ everywhere. Again, we are interested in estimation of the principle eigenvalues, which consist of the typical, well-known Sturm-Liouville eigenvalue problem. Refer to Egorov & Kondratiev (1996) for the present status of the study and references. Here, we mention two results, which are the most general ones we have ever known before.

Theorem 3.1. *Let* $b(x) \equiv 0$ *(which corresponds to the birth-death process with* $a_i = b_i$ *for all* $i \geq 1$*) and set* $\delta = \sup_{x>0} x \int_x^\infty a^{-1}$*. Here we omit the integration variable when it is integrated with respect to the Lebesgue measure. Then, we have*

1. *Kac & Krein (1958):* $\delta^{-1} \geq \lambda_0 \geq (4\delta)^{-1}$*, here* λ_0 *is the first eigenvalue corresponding to the Dirichlet boundary* $f(0) = 0$*.*
2. *Kotani & Watanabe (1982):* $\delta^{-1} \geq \lambda_1 \geq (4\delta)^{-1}$*.*

In the diffusion context, we also obtained a variational formula.

Theorem 3.2. *(Chen & Wang (1997a), Chen(2000b)).*

$$\lambda_1 = \sup_{f\in\mathcal{F}} \inf_{x>0} \left[\frac{e^{-C(x)}}{f'(x)} \int_x^\infty \frac{f(u)e^{C(u)}}{a(u)} \mathrm{d}u\right]^{-1}. \tag{3.1}$$

where

$$C(x) = \int_0^x b/a, \quad \pi(\mathrm{d}x) = \frac{1}{Z}\frac{e^{C(x)}}{a(x)}\mathrm{d}x \ \ \textit{(Z is the normalizing constant) and}$$
$$\mathcal{F} = \{f \in L^1(\pi) : \pi(f) \geq 0 \ \textit{and} \ f'|_{(0,\infty)} > 0\}.$$

In the papers quoted above, it was proved that the equality in (3.1) replaced by "$\geq$" holds in general and the equality holds whenever both a and b are continuous. Clearly, the equality should also hold for certain measurable a and b, in virtue of a standard approximating procedure and (3.2) below. Again, it is unclear from the result whether there exists a representative function $f \in \mathcal{F}$ or not to justify the positiveness of λ_0 or λ_1.

Before moving further, let us recall the classical variational formula for λ_1:

$$\lambda_1 = \inf\{D(f) : \pi(f) = 0, \pi(f^2) = 1\}, \tag{3.2}$$
$$\pi(\mathrm{d}x) = e^{C(x)}\mathrm{d}x/(a(x)Z), \quad D(f) = \int_0^\infty a f'^2 \mathrm{d}\pi.$$

Note that there is no common point between (3.1) and (3.2). The formula (3.1) (resp. (3.2)) is especially meaningful in estimating the lower (resp. upper) bounds of λ_1. Now, it is simple matter to rewrite (3.2) as (3.3) below. Similarly, we have (3.4) for λ_0.

Poincaré inequalities.

$$\lambda_1 : \quad \mathrm{var}(f) \leq \lambda_1^{-1} D(f) \tag{3.3}$$
$$\lambda_0 : \quad \|f\|^2 \leq \lambda_0^{-1} D(f), \quad f(0) = 0. \tag{3.4}$$

It is interesting that inequality (3.4) is a special but typical case of the weighted Hardy inequality discussed in the next section.

4. Weighted Hardy inequality. The classical Hardy inequality goes back to Hardy (1920):

$$\int_0^\infty \left(\frac{f}{x}\right)^p \leq \left(\frac{p}{p-1}\right)^p \int_0^\infty f'^p, \quad f(0) = 0, f' \geq 0,$$

where the optimal constant was determined by Landau (1926). After a long period of efforts by analysts, the inequality was finally extended to the following form.

Weighted Hardy inequality (Muckenhoupt (1972)). *Let ν and λ be nonnegative Borel measures. Then*

$$\int_0^\infty f^2 \mathrm{d}\nu \leq A \int_0^\infty f'^2 \mathrm{d}\lambda, \quad f \in C^1, f(0) = 0, \tag{4.1}$$

where $B = \sup_{x>0} \nu[x,\infty] \int_x^\infty (\mathrm{d}\lambda_{\mathrm{abs}}/\mathrm{dLeb})^{-1}$ satisfies $B \leq A \leq 4B$ and $\mathrm{d}\lambda_{\mathrm{abs}}/\mathrm{dLeb}$ is the derivative of the absolutely continuous part of λ with respect to the Lebesgue measure.

The Hardy-type inequalities play a very important role in the study of harmonic analysis and have been treated in many publications. Refer to the books: Opic & Kufner (1990), Dynkin (1990), Mazya (1985) and the survey article Davies (1999) for more details. The author learnt the weighted inequality from Miclo (1999a), Bobkov & Götze (1999a, b).

It is now easy to deduce the Poincaré inequality (3.4) from (4.1), simply setting $\nu = \pi$ and $\lambda = e^C \mathrm{d}x$. In other words, Muckenhoupt's Theorem $\Longrightarrow$ Kac & Krein's Theorem.

Thanks are given to the weighted Hardy inequality, from which we learnt that there must exist a representative test function. Then, it is not difficult to figure out that the function is simply $f(x) = \sqrt{\int_0^x e^{-C}}$.

Theorem 4.1 (Chen (2000b, 2001a)). *Let $\delta = \sup_{x>0} \int_0^x e^{-C} \int_x^\infty e^C/a$. Then, we have*

(1) ${\delta'}^{-1} \geq \lambda_0 \geq (4\delta)^{-1}$, *where δ' is an explicit constant satisfying $\delta \leq \delta' \leq 2\delta$. Moreover, λ_0 (resp. λ_1)> 0 iff $\delta < \infty$.*

(2) $$\lambda_0 = \sup_{f \in \mathcal{F}} \inf_{x>0} I(f)(x)^{-1}$$
$$= \inf_{f \in \mathcal{F}'} \sup_{x>0} I(f)(x)^{-1} \text{ (Completed variational formula!)}.$$

Here, the set $\mathcal{F}'$ is a modification of $\mathcal{F}$ (The readers are urged to find out the details from the original papers). The lower bound in part (1) is an application of (3.1) to the representative function mentioned above. The upper bound comes from a direct computation. Part (1) improves (4.1) in the present situation. Clearly, part (2) is much finer result than (4.1) and can not be deduced from (4.1). The second line in part (2) is a natural dual of the first one but is completely different from the classical one (3.4).

The result can be immediately applied to the whole line or higher-dimensional situation. For instance, for Laplacian on compact Riemannian manifolds, it was proved by Chen & Wang (1997b) that

$$\lambda_1 \geq \sup_{f\in\mathcal{F}} \inf_{r\in(0,D)} I(f)(r)^{-1} =: \xi_1$$

for some $I(f)$, which is similar to the right-hand side of (3.1). We now have ${\delta'}^{-1} \geq \xi_1 \geq (4\delta)^{-1}$ for some constants δ and δ' similar to Theorem 4.1. Refer to Chen (2000b) for details.

We now return to birth-death processes. A parallel result of Theorem 4.1 has been presented in Chen (2000b). In particular, we obtain a criterion for the positiveness of λ_1 for birth-death processes. This is also done by Miclo (1999b) in terms of discrete Hardy inequalities. Combining this with Theorem 2.1, we finally obtain a criterion for the exponential ergodicity of birth-death processes. The result is included in Table 5.1 below. We mention that it is now possible to prove the last criterion directly from (1.5), based on the study on λ_1. However, the more general case remains open.

Open problem 2. *Does there exist a criterion for exponential ergodicity of single-birth processes?*

5. Three basic inequalities. Denote by var(f)= $\|f-\pi(f)\|^2 = \pi(f^2)-\pi(f)^2$. Then, the inequality (3.3) can be rewritten as (5.1) below. On the other hand, one may study other inequalities, for instance, the logarithmic or Nash inequality listed below.

$$\textit{Poincaré inequality}: \quad \operatorname{var}(f) \leq \lambda_1^{-1} D(f) \tag{5.1}$$

$$\textit{LogS inequality}: \quad \int f^2 \log(|f|/\|f\|)\mathrm{d}\pi \leq \sigma^{-1} D(f) \tag{5.2}$$

$$\textit{Nash inequality}: \quad \operatorname{var}(f)^{1+2/\nu} \leq \eta^{-1} D(f)\|f\|_1^{4/\nu} \quad \text{(for some } \nu>0). \tag{5.3}$$

Here, to save notation, σ (resp. η) denotes the largest constant so that (5.2) (resp. (5.3)) holds.

Each inequality describes a type of ergodicity. First, (5.1) $\Longleftrightarrow$ (1.6). Next, the logarithmic Sobolev inequality implies the decay of the semigroup $P(t)$ to π exponentially in relative entropy with rate σ and the Nash inequality is equivalent to $\operatorname{var}(P(t)(f)) \leq C\|f\|_1/t^{\nu/2}$. For reversible Markov chains, a complete diagram of these types of ergodicity was presented in Chen (1999, 2001b).

Fortunately, the criteria (also based on the weighted Hardy inequality) for the last two inequalities as well as for the discrete spectrum (which means that there is no continuous spectrum and moreover, all eigenvalues have finite multiplicity) are obtained by Mao (2000a,b,c). We can now summarize the results in Table 5.1. The table is arranged in such order that the property in the latter line is stranger than the former one, the only exception is that even though the strong ergodicity is often stronger than the logarithmic Sobolev inequality but they are not comparable in general (Chen (2001b)).

Birth-death processes

$$\begin{aligned} i \to i+1 \quad &\text{at rate} \quad b_i = q_{i,i+1} > 0\\ \to i-1 \quad &\text{at rate} \quad a_i = q_{i,i-1} > 0. \end{aligned}$$

Define

$$\mu_0 = 1, \quad \mu_n = \frac{b_0\cdots b_{n-1}}{a_1\cdots a_n},\ n\geq 1; \quad \mu[i,k] = \sum_{i\leq j\leq k} \mu_j.$$

Property	Criterion
Uniqueness	$\sum_{n\geq 0}\frac{1}{\mu_n b_n}\mu[0,n]=\infty \quad (*)$
Recurrence	$\sum_{n\geq 0}\frac{1}{\mu_n b_n}=\infty$
Ergodicity	$(*)$ & $\mu[0,\infty)<\infty$
Exponential ergodicity	$(*)$ & $\sup_{n\geq 1}\mu[n,\infty)\sum_{j\leq n-1}\frac{1}{\mu_j b_j}<\infty$
Discrete spectrum	$(*)$ & $\lim_{n\to\infty}\sup_{k\geq n+1}\mu[k,\infty)\sum_{j=n}^{k-1}\frac{1}{\mu_j b_j}=0$
LogS inequality	$(*)$ & $\sup_{n\geq 1}\mu[n,\infty)\log[\mu[n,\infty)^{-1}]\sum_{j\leq n-1}\frac{1}{\mu_j b_j}<\infty$
Strong ergodicity	$(*)$ & $\sum_{n\geq 0}\frac{1}{\mu_n b_n}\mu[n+1,\infty)=\sum_{n\geq 1}\mu_n\sum_{j\leq n-1}\frac{1}{\mu_j b_j}<\infty$
Nash inequality	$(*)$ & $\sup_{n\geq 1}\mu[n,\infty)^{(\nu-2)/\nu}\sum_{j\leq n-1}\frac{1}{\mu_j b_j}<\infty(\varepsilon)$

Table 5.1

Here, "$(*)$ & $\cdots$" means that one requires the uniqueness condition in the first line plus the condition "$\cdots$". The "(ε)" in the last line means that there is still a small gap from being necessary.

Open problem 3. *Find out a criterion for the Nash inequality in dimension one.*

In view of Theorem 4.1, it is natural to ask the following

Open question 4. *Does there exist a variational formula in the form of part* (2) *of Theorem* 4.1 *for the logarithmic Sobolev or Nash inequality?*

Diffusion processes on $[0,\infty)$

$$L=a(x)\frac{\mathrm{d}^2}{\mathrm{d}x^2}+b(x)\frac{\mathrm{d}}{\mathrm{d}x}.$$

Define

$$C(x)=\int_0^x b/a,\qquad \mu[x,y]=\int_x^y e^C/a.$$

Property	Criterion
Uniqueness	$\int_0^\infty \mu[0,x]e^{-C(x)}=\infty \quad (*)$
Recurrence	$\int_0^\infty e^{-C(x)}=\infty$
Ergodicity	$(*)$ & $\mu[0,\infty)<\infty$
Poincaré inequality	$(*)$ & $\sup_{x>0}\mu[x,\infty)\int_0^x e^{-C}<\infty$
Discrete spectrum	$(*)$ & $\lim_{n\to\infty}\sup_{x>n}\mu[x,\infty)\int_n^x e^{-C}=0$
LogS inequality	$(*)$ & $\sup_{x>0}\mu[x,\infty)\log[\mu[x,\infty)^{-1}]\int_0^x e^{-C}<\infty$
Strong ergodicity	$(*)$ & $\int_0^\infty \mu[x,\infty)e^{-C(x)}<\infty?$
Nash inequality	$(*)$ & $\sup_{x>0}\mu[x,\infty)^{(\nu-2)/\nu}\int_0^x e^{-C}<\infty(\varepsilon)$

Table 5.2

For Nash inequality, we have the same remark as before. Note that the exponential ergodicity in Table 5.1 is replaced by Poincaré inequality here. This suggests the following problem which is conjectured to be true (partially solved in Chen (1998, 2000a). Here we mention a simple fact which was missed in the original papers. In the reversible case, for $p_s(x, y) := \mathrm{d}P_s(x, \cdot)/\mathrm{d}\lambda$, we have $\int (p_s(x, y)/\pi(y))^2 \pi(\mathrm{d}y) = \int p_s(x, y)p_s(y, x)\lambda(\mathrm{d}y)/\pi(x) = p_{2s}(x, x)/\pi(x) < \infty$. Hence $p_s(x, \cdot)/\pi \in L^2(\pi)$).

Open problem 5. *Prove that the exponentially ergodic convergence rate coincides with the L^2-exponential convergence rate, or at least these two convergences are equivalent.*

The question marked in the line of strong ergodicity means that it remains unproved, but conjectured to be true in terms of the similarity between one-dimensional diffusions and birth-death processes.

Open problem 6. *Prove the criterion for strong ergodicity listed in Table 5.2.*

Added in Proof. A large part of the open problems, except 3 and 4, have been solved by Yong-Hua Mao and Yu-Hui Zhang in the past few months.

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